Transience, Density, and Point Recurrence of Multiparameter Brownian Motion

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The transience, density, and point recurrence properties of one-parameter Brownian motion have been known for some time. If the Brownian path is in three-dimensional Euclidean space, the path is nowhere dense with probability one and goes to infinity as the time parameter goes to infinity. In d-dimensional space, $d \geqq 5$, the probability is one that the path has no double points [2]. (A different proof [1] shows there are no double points in 4-space.)

The original proofs of these properties depend on Lévy's equality:

$$(1) \qquad P\{\max_{t \varepsilon [a,b]} X(t, \omega) - X(a, \omega) > \lambda\} = 2P\{X(b, \omega) - X(a, \omega) > \lambda\}.$$

This equality requires independent increments, which do not exist in general for multiparameter Brownian motion. It turns out that Lévy's equality is not needed. The non-recurrence property follows from Lévy's result on the modulus of continuity of multiparameter Brownian motion [3]. The transience and density properties follow from an inequality given here as Theorem 2. These proofs seem to be new, even for the one-parameter case.

Let $W^{(N,d)}$ denote Lévy's N-parameter Brownian motion with values in d-dimensional Euclidean space; *i.e.* if $X = W^{(N,d)}$, then $X(t, \omega) = (X_1(t, \omega), \cdots, X_d(t, \omega)) \varepsilon \mathbf{R}^d$, where $t = (t_1, \cdots, t_N) \varepsilon \mathbf{R}^N$ and the coordinate functions X_i are mutually independent, separable, Gaussian processes with mean zero and covariance

$$E(X_i(s), X_i(t)) = (1/2)[|s| + |t| - |s - t|].$$

Here $|\cdot|$ is the Euclidean norm.

Theorem 1. *If* $4N < d$, *then almost no paths of* $W^{(N,d)}$ *have double points*; i.e. $P\{X(s, \omega) = X(t, \omega)$ *for some distinct* $s, t\} = 0$.

Proof. We use Lévy's modulus of continuity [3], which shows that if $0 < \alpha < 1/2$ and if A is a bounded set in $\mathbf{R}^N$, then

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$$(2) \qquad P\{\limsup_{\substack{s,t\in A\\ |s-t|\to 0}} |X_p(s) - X_p(t)|\cdot|s - t|^{-\alpha} = 0\} = 1, \qquad p = 1, \cdots, d.$$

Let A, B be cubes in $\mathbf{R}^N$. Assume $d(A, B)$, the distance from A to B, is positive. It is sufficient to show that

$$\beta = P\{X(s, \omega) = X(t, \omega) \quad \text{for some} \quad s \in A, t \in B\} = 0.$$

Choose α so that $4N < 2d\alpha < d$. Partition A and B into 2^{nN} cubes by dividing the edges of each cube into 2^n equal segments. Call the cubes A_i and B_i, $i = 1, \cdots, 2^{nN}$, each with a vertex s_i, t_i, respectively. Then

$$\beta \leqq \sum_{i=1}^{2^{nN}} \sum_{j=1}^{2^{nN}} P\{|X_p(s_i) - X_p(t_j)| < 2\cdot 2^{-n\alpha}, \quad p = 1, \cdots, d\}$$
$$+ \sum_{p=1}^{d} P\{\max_{s\in A_i} |X_p(s) - X_p(s_i)| > 2^{-n\alpha} \quad \text{for some} \quad i = 1, \cdots, 2^{nN}\}$$
$$+ \sum_{p=1}^{d} P\{\max_{t\in B_j} |X_p(t) - X_p(t_j)| > 2^{-n\alpha} \quad \text{for some} \quad j = 1, \cdots, 2^{nN}\}.$$

Equation (2) shows that the last two sums go to zero as n goes to infinity. The first sum is bounded by

$$\sum_{i=1}^{2^{nN}} \sum_{j=1}^{2^{nN}} \left\{(2\pi\, |s_i - t_j|)^{-1/2} \int_{-2\cdot 2^{-n\alpha}}^{2\cdot 2^{-n\alpha}} \exp\left[-u^2/2\, |s_i - t_j|\right] du\right\}^d.$$

This is bounded by $[d(A, B)]^{-d/2}\cdot 2^{n(2N-\alpha d)}$, which also goes to zero as n goes to infinity.

The next theorem is needed as a substitute for equation (1) in order to prove Theorem 3. The proof is an amplification of a classic argument of Lévy [3], [4].

Let A be a cube in $\mathbf{R}^N$ with D the length of an edge. Dividing each edge into 2^n equal segments partitions A into 2^{nN} cubes. Call these cubes A_i, $i = 1, \cdots, 2^{nN}$, with t_i a vertex of A_i.

Theorem 2. *For* $X = W^{(N,1)}$, *there are constants (i.e. independent of* n *and* D*)* $K_1 > 0$, $K_2 > 0$, *and* $0 < a < 1$ *such that*

$$P\{\max_{t\in A_i} |X(t, \omega) - X(t_i, \omega)| > K_1(D2^{-n} \log 2^n)^{1/2} \text{ for some } i\} < K_2 a^n.$$

Proof. Since X is continuous with probability one [3], it is sufficient to prove the theorem on the binary rational points in A. We may also assume A is oriented with the axes and has one vertex at the origin. Let $m = 2^\nu$ and consider the points in A whose coordinates are integral multiples of D/m. These points form a network of $Nm(m + 1)^{N-1}$ equal segments parallel to the axes, each of length D/m. Call this network π_ν. For the increment ΔX on each of these segments,

$$P\{|\Delta X| > \lambda m^{-1/2}\} \leqq D^{1/2}\lambda^{-1} \exp - (\lambda^2/2D).$$

Let

$$\alpha_\nu = P\{|\Delta X| > \lambda m^{-1/2} \text{ on some segment in } \pi_\nu\}.$$

Then

$$\alpha_\nu \leqq Nm(m+1)^{N-1}D^{1/2}\lambda^{-1} \exp - (\lambda^2/2D).$$

Let $C > 1$, $\lambda_\nu = C(2DN\nu \log 2)^{1/2}$, and let

$$B_n = \{\omega: |\Delta X| > \lambda_\nu m^{-1/2} \text{ on some segment in some } \pi_\nu\,,\ \nu \geqq n\}.$$

Then

$$P(B_n) \leqq \sum_{\nu=n}^{\infty} \alpha_\nu \leqq \sum_{\nu=n}^{\infty} N2^\nu(2^\nu+1)^{N-1}\, D^{1/2}\lambda_\nu^{-1} \exp - (\lambda^2/2D)$$

$$\leqq K \sum_{\nu=n}^{\infty} [2^{N(1-C^2)}]^\nu = K\frac{[2^{N(1-C^2)}]^n}{1-2^{N(1-C^2)}} = K_2 a^n.$$

The proof is completed by showing there is a constant $K_1 > 0$ such that for each $\omega \in B_n^C$ = the complement of B_n and each $i = 1, \cdots, 2^{nN}$,

$$\max_{t \in A_i} |X(t,\ \omega) - X(t_i\,, \omega)| \leqq K_1(D2^{-n} \log 2^n)^{1/2}.$$

It suffices to suppose $s, t \in A$ are on a line parallel to one axis since any segment can be decomposed into lines parallel to the axes. Say the p^{th} component of s is $s_p = D\,q/2^r$, where $r \geqq n$ and $0 \leqq q \leqq 2^r$. Suppose $t = s + De_p \sum_{i=1}^{u} \epsilon_i/2^{r+i}$, where e_p is the unit vector parallel to the p^{th} axis and ϵ_i is 0 or 1. Note that $\omega \in B_n^C$ means that $|\Delta X| < \lambda_\nu 2^{-1/2}$ for every increment ΔX in every network π_ν, $\nu \geqq n$. Then $|X(s, \omega) - X(t, \omega)| \leqq \sum_{i=1}^{u} \epsilon_i \lambda_{r+1} 2^{-(r+i)/2}$. Let j be the first integer such that $\epsilon_j = 1$. If we let $i' = i - j + 1$, then $i \geqq j$ and a computation gives

$$r + i \leqq (r+j)i'.$$

Now

$$|X(s) - X(t)| \leqq C(2DN \log 2)^{1/2} \sum_{i=j}^{u-j+1} \epsilon_i[(r+i)2^{-(r+i)}]^{1/2}$$

$$\leqq K(D \log 2)^{1/2} \sum_{i'=1}^{h-j+1} [(r+j)2^{-(r+i)}]^{1/2}[i'2^{1-i'}]^{1/2}$$

$$\leqq K_1(D \log 2^{r+i}/2^{r+i})^{1/2} \leqq K_1(D2^{-n} \log 2^n)^{1/2}.$$

Theorem 3. *If $2N < d$, the probability is one that the path of $W^{(N,d)}$ goes to infinity as the parameter goes to infinity; i.e. for almost every ω, given $M > 0$ there is an $M(\omega)$ such that $|t| > M(\omega)$ implies $|X(t, \omega)| > M$.*

Proof. Choose α so that $2N < \alpha d < d$. Let $a_k = k^\alpha$, $k = 1, 2, \cdots$. There is a constant $c > 0$ such that

$$ck^{\alpha-1} < a_{k+1} - a_k < k^{\alpha-1}.$$

Let $d_k = a_{k+1} - a_k$ and $n_k = [a_k/d_k]$ = the integer part of a_k/d_k. Then $n_k < k/c$.

Cover $\mathbf{R}^N$ with cubes of the form

$$[\pm i_1 d_k, \pm(i_1 + 1)d_k] \times \cdots \times [\pm i_N d_k, \pm(i_N + 1)d_k]$$

with one of the indices i fixed at n_k, while the other indices are equal to any integer from 0 to n_k.

For each k the edge length of each cube is d_k and the number of cubes is $b_k = 2^N N(n_k + 1)^{N-1} < Kk^{N-1}$. Call the cubes A_{ki}, $i = 1, \cdots, b_k$, each with a vertex t_{ki}.

Now let $M > 0$. (Assume M is large enough that $2^N \exp - (M^2/2N^{1/2}) < 1$ and $K_1(2^{-n} \log 2^n)^{1/2} < M$ for all n.) Let $C_k = [k^{1-\alpha}]$ = the integer part of $k^{1-\alpha}$ and partition A_{ki} into $2^{C_k N}$ cubes by dividing each edge of A_{ki} into 2^{C_k} equal segments. Call these A_{kij}, $j = 1, \cdots, 2^{C_k}$, each with a vertex t_{kij}.

Let

$$\beta_{ki} = P\{|X_p(t)| < M, p = 1, \cdots, d, \quad \text{for some } t \in A_{ki}\}.$$

Then

$$\begin{aligned}
\beta_{ki} &\leqq P\{|X_p(t_{ki})| < 3M, \quad p = 1, \cdots, d\} \\
&\quad + \sum_{p=1}^{d} \sum_{j=1}^{2^{C_k N}} P\{|X_p(t_{kij}) - X_p(t_{ki})| > M\} \\
&\quad + \sum_{p=1}^{d} P\{\max_{t \in A_{kij}} |X_p(t) - X_p(t_{kij})| > M \quad \text{for some} \quad j = 1, \cdots, 2^{C_k N}\} \\
&\leqq \left\{(2\pi\, |t_{ki}|)^{-1/2} \int_{-3M}^{3M} \exp\left[\frac{-u^2}{2\,|t_{ki}|}\right] du\right\}^d \\
&\quad + d2^{C_k N} P\{|X_1(s_{ki}) - X_1(t_{ki})| > M, \quad s_{ki} \text{ the vertex opposite } t_{ki}\} + dK_2 a^{C_k} \\
&\leqq [6Ma_k^{-1/2}]^d + d2^{C_k N}(d_k N^{1/2})^{1/2} M^{-1} \exp\left[\frac{-M^2}{2d_k N^{1/2}}\right] + dK_2 a^{C_k} \\
&\leqq K_3\left\{k^{-\alpha d/2} + \left[2^N \exp\left(\frac{-M^2}{2N^{1/2}}\right)\right]^{k^{1-\alpha}} + a^{k^{1-\alpha}}\right\}.
\end{aligned}$$

Summing all the β_{ki},

$$\sum_{k=1}^{\infty} \sum_{i=1}^{b_k} \beta_{ki} \leqq \sum_{k=1}^{\infty} K_4 k^{N-1}\left\{k^{-\alpha d/2} + \left[2^N \exp\left(\frac{-M^2}{2N^{1/2}}\right)\right]^{k^{1-\alpha}} + a^{k^{1-\alpha}}\right\}.$$

Since this is a convergent series, the theorem follows from the Borel–Cantelli lemma.

Theorem 4. *If $2N < d$, the range of $W^{(N,d)}$ is nowhere dense in $\mathbf{R}^d$ with probability one.*

Proof. The range has Lebesgue measure zero with probability one. To show this, divide the unit cube in $\mathbf{R}^N$ into h-cubes (cubes with edge of length h). Choose δ so that $N/d < \delta < 1/2$. By equation (2) the image of each h-cube is contained in a set of diameter Kh^δ, and the estimate of the d measure of the image is $h^{-N}(Kh^\delta)^d = K^d h^{\delta d - N}$, which goes to zero as h goes to zero.

Theorem 3 shows that the range is closed with probability one. This fact and the zero Lebesgue measure show the range is nowhere dense.

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REFERENCES

1. A. DVORETZKY, P. ERDÖS & S. KAKUTANI, *Double points of paths of Brownian motion in n-space*, Acta Szeged **12** (1950), 75–81.
2. S. KAKUTANI, *On Brownian motions in n-space*, Japan Acad. Tokyo Proc. **20** (1944), 648–652.
3. P. LÉVY, *Processus stochastiques et mouvement Brownien.* Paris, 1948, p. 265.
4. P. LÉVY, *Théorie de l'addition des variables aléatoires.* Paris, 1954, pp. 169–172.

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