

## FUNCTIONS WITH DIFFERENT STRONG AND WEAK $\Phi$ -VARIATIONS

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**ABSTRACT.** This paper shows by example how different the strong  $\Phi$ -variation can be from the weak  $\Phi$ -variation. Let  $\Phi$  be a convex function on  $[0, \infty)$  with  $\Phi(0) = 0$ . A continuous function  $f$  on  $[a, b]$  is of bounded strong  $\Phi$ -variation if  $\sup \sum \Phi(|f(x_i) - f(x_{i-1})|) < \infty$ , for the partitions of  $[a, b]$ . Since  $\inf \sum \Phi(|f(x_i) - f(x_{i-1})|) = 0$  if  $\lim_{x \rightarrow 0} x^{-1} \Phi(x) = 0$ , the weak  $\Phi$ -variation is defined as  $\inf \sum \Phi(\omega(f; x_{i-1}, x_i))$ , where  $\omega(f; c, d)$  is the oscillation of  $f$  on  $[c, d]$ . Of special interest is the case  $\Phi(x) = x^p$ ,  $p \geq 1$ , in terms of which strong and weak variation dimensions are defined. They are denoted by  $\dim_s(f)$  and  $\dim_w(f)$ , respectively. By a lemma of Goffman and Loughlin, the Hausdorff dimension  $d$  of the graph of  $f$  provides a lower bound for  $\dim_w(f)$ :  $1/(2-d) \leq \dim_w(f)$ . A Lipschitz condition of order  $\alpha$  provides an upper bound for  $\dim_s(f)$ :  $\dim_s(f) \leq 1/\alpha$ . Besicovitch and Ursell showed that  $1 \leq d \leq 2 - \alpha$  and gave examples to show that  $d$  can take on any value in this interval. It turns out that these examples provide the extreme cases for variation dimensions; i.e.,  $\dim_w(f) = 1/(2-d)$  and  $\dim_s(f) = 1/\alpha$ .

**1. Introduction.** The purpose of this paper is to show by example how different the strong variation of a function can be from the weak variation. In previous work concerning the variation of Brownian motion, the strong and weak variation dimensions were shown to be equal.

Strong and weak  $\Phi$ -variations were defined by Goffman and Loughlin [2]. If  $\Phi$  is a nonnegative convex function on  $[0, \infty)$  with  $\Phi(0) = 0$ , and  $f$  is a real function on  $[a, b]$ , the strong  $\Phi$ -variation of  $f$  is

$$V_{\Phi}^s(f) = \sup_{\pi} \sum_{i=1}^n \Phi(|f(x_i) - f(x_{i-1})|),$$

where  $\pi$  is any partition  $a = x_0 < \dots < x_n = b$  of  $[a, b]$ . As Goffman and Loughlin pointed out, if  $\lim_{x \rightarrow 0} x^{-1} \Phi(x) = 0$ , then, for every continuous  $f$ ,

$$\inf_{\pi} \sum_{i=1}^n \Phi(|f(x_i) - f(x_{i-1})|) = 0.$$

This fact motivates the definition of weak  $\Phi$ -variation,

$$V_{\Phi}^w(f) = \inf_{\pi} \sum_{i=1}^n \Phi(\omega(f; x_{i-1}, x_i)),$$

where

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$$\omega(f; c, d) = \sup_{c \leq x < y \leq d} |f(x) - f(y)|.$$

Using  $\Phi(x) = x^p, p \geq 1$ , the  $\Phi$ -variations of  $f$  will be called the strong  $p$ -variation,  $V_p^s(f)$ , and the weak  $p$ -variation,  $V_p^w(f)$ . Then either  $V_p^s(f) = \infty$  for all  $p \geq 1, V_p^s(f) < \infty$  for all  $p > 1$ , or there is a unique  $p > 1$  such that  $V_q^s(f) = \infty$  for all  $q < p$  and  $V_q^s(f) < \infty$  for all  $q > p$ . This yields the strong variation dimension of  $f, \dim_s(f)$ , which is  $\infty, 1$ , and  $p$ , respectively, in the three cases. Similarly, the weak variation dimension of  $f, \dim_w(f)$  is defined according to  $V_p^w(f) > 0$  or  $V_p^w(f) = 0$ .

*Note.* Goffman and Loughlin defined the weak  $\Phi$ -variation as  $\liminf \sum \Phi(\omega(f; x_{i-1}, x_i))$  as the norms of the partitions converge to zero. The weak  $\Phi$ -variation given here leaves  $\dim_w(f)$  unchanged and was so defined simply because it more closely parallels the definition of strong variation.

Goffman and Loughlin showed that both the strong and weak variation dimensions of Brownian motion are two with probability one. At the suggestion of Professor Goffman I extended the concepts of strong and weak variation to higher dimensions and showed that with appropriate definitions,  $N$ -parameter Brownian motion in  $d$ -space has strong and weak variation dimensions  $2N$  with probability one [4], [5]. Taylor [3] found precise functions  $\Phi$  for measuring the strong and weak  $\Phi$ -variations of Brownian motion.

The Lipschitz condition of a function gives an upper bound for the strong variation dimension, and the Hausdorff dimension of the graph gives a lower bound for the weak variation dimension. The examples given here, taken from Besicovitch and Ursell [1], are the extreme cases; that is, they have the largest possible strong variation dimension and the smallest possible weak variation dimension.

**2. Preliminary lemmas.** It is clear from the definitions that for every function  $f, V_p^w(f) \leq V_p^s(f)$ , and therefore

$$(1) \quad \dim_w(f) \leq \dim_s(f).$$

A function  $f$  on  $[0, 1]$  satisfies a Lipschitz condition of order  $\delta (f \in \text{Lip } \delta)$  if

$$M_f = \sup_{x \neq y} |f(x) - f(y)| |x - y|^{-\delta} < \infty.$$

The following lemma shows that if  $f \in \text{Lip } \delta$ , then

$$(2) \quad \dim_s(f) \leq 1/\delta.$$

**LEMMA 1.** *If  $f \in \text{Lip } \delta$ , then  $V_{1/\delta}^s(f) < \infty$ .*

**PROOF.** For any partition  $0 = x_0 < x_1 < \dots < x_n = 1$ ,

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})|^{1/\delta} \leq \sum_{i=1}^n [M_f(x_i - x_{i-1})^\delta]^{1/\delta} = M_f^{1/\delta}.$$

The following lemma, due to Goffman and Loughlin [2], shows that if the Hausdorff dimension of the graph of  $f$  is  $d$ , then

$$(3) \quad 1/(2 - d) \leq \dim_w(f).$$

LEMMA 2. If  $V_p^w(f) = 0$ , then  $H_{2-1/p}(\text{gr } f) = 0$ . (Here  $H_k$  is the Hausdorff  $k$  dimensional measure and  $\text{gr } f$  is the graph of  $f$ .)

We note here that Goffman and Loughlin stated the lemma in a different form, but the different definition of weak  $\Phi$ -variation does not affect the validity of Lemma 2. All that is required for the proof of the lemma is a sequence of partitions of  $[a, b]$ , with norms converging to zero, which gives  $V_p^w(f)$ . Under the present definition of  $V_p^w(f)$ , if  $V_p^w(f) = 0$ , we have such a sequence.

Let

$$E_f(\delta, M) = \left\{ x \in [0, 1]: \limsup_{y \rightarrow x} |f(x) - f(y)| |x - y|^{-\delta} \geq M \right\}.$$

LEMMA 3. If  $E_f(\delta, M)$  has positive Lebesgue measure for some  $M > 0$ , then  $\dim_s(f) \geq 1/\delta$ .

PROOF. Suppose  $|E_f(\delta, M)| > 0$  for some  $M > 0$  and let  $p < 1/\delta$ . It is sufficient to show  $V_p^s(f) = \infty$ . Let  $\epsilon > 0$ . For each  $x \in E(\delta, M)$  there is an  $h_x < \epsilon$  such that

$$|f(x) - f(x + h_x)| h_x^{-\delta} \geq M/2.$$

The intervals  $[x, x + h_x]$  cover  $E_f(\delta, M)$  in the sense of Vitali, so there is a finite disjoint collection  $\{[x_1, x_1 + h_1], \dots, [x_N, x_N + h_N]\}$  such that

$$\left| E_f(\delta, M) \sim \bigcup_{i=1}^N [x_i, x_i + h_i] \right| < \frac{1}{2} |E_f(\delta, M)|.$$

Now

$$\begin{aligned} V_p^s(f) &\geq \sum_{i=1}^N |f(x_i) - f(x_i + h_i)|^p > \sum_{i=1}^N h_i^{\delta p} \left(\frac{M}{2}\right)^p \\ &> \left(\frac{M}{2}\right)^p \epsilon^{\delta p - 1} \sum_{i=1}^N h_i > \left(\frac{M}{2}\right)^p \epsilon^{\delta p - 1} \frac{1}{2} |E_f(\delta, M)|, \end{aligned}$$

which goes to  $\infty$  as  $\epsilon \rightarrow 0$ .

3. **The functions of Besicovitch and Ursell.** Besicovitch and Ursell [1] showed that if  $f \in \text{Lip } \delta$ , then the Hausdorff dimension  $d$  of the graph of  $f$  satisfies  $1 \leq d \leq 2 - \delta$ , and gave examples which showed that any  $d$  in this interval is possible.

The function  $f$  is defined as follows. Let  $\varphi(x) = 2x$  for  $0 \leq x \leq \frac{1}{2}$ , and elsewhere define  $\varphi$  by  $\varphi(x) = \varphi(-x) = \varphi(x + 1)$ . Define  $f$  by

$$f(x) = \sum_{n=1}^{\infty} a_n \varphi(b_n x)$$

where  $a_n = b_n^{-\delta}$  and  $b_{n+1} = b_n^{\mu}$ .

For  $\mu$ , consider three cases.

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(i) For  $1 < d < 2 - \delta$ ,  $\mu_n = \mu = (1 - \delta)\delta^{-1}(2 - d)(d - 1)^{-1}$ .

- (ii) For  $d = 2 - \delta$ ,  $\mu_n$  is chosen so that  $\mu_n \rightarrow 1$  and  $b_{n+1}/b_n \rightarrow \infty$ . (For example,  $b_1 > 1$  and  $\mu_n = 1 + n^{-1/2}$ .)
  - (iii) For  $d = 1$ ,  $\mu_n$  is chosen so that  $b_n \rightarrow \infty$  and  $\mu_n \rightarrow \infty$ .
- In (i) and (ii) we have

$$(4) \quad \mu_n \geq \mu = (1 - \delta)\delta^{-1}(2 - d)(d - 1)^{-1}.$$

In (iii) we may assume

$$(5) \quad \mu_n \geq (1 - \delta)\delta^{-1}(2 - d_n)(d_n - 1)^{-1}, \quad \text{where } d_n \rightarrow 1.$$

In all three cases  $b_1$  may be chosen so that

$$(6) \quad b_{n+1} \geq Bb_n, \quad \text{where } B > 1.$$

**LEMMA 4.** *For the function  $f$  constructed above, there is an  $M > 0$  such that  $E_f(\delta, M) = [0, 1]$ .*

**PROOF.** For any  $x \in [0, 1]$ ,

$$\Delta f = f(x + h) - f(x) = \sum a_n[\varphi(b_n x + b_n h) - \varphi(b_n x)].$$

Let

$$s_\nu = \sum_{n \leq \nu} a_n[\varphi(b_n x + b_n h) - \varphi(b_n x)]$$

and

$$r_\nu = \sum_{n > \nu} a_n[\varphi(b_n x + b_n h) - \varphi(b_n x)].$$

Then

$$\begin{aligned} |r_\nu| &\leq \sum_{n > \nu} a_n |\varphi(b_n x + b_n h) - \varphi(b_n x)| \leq \sum_{n > \nu} a_n = \sum_{n > \nu} b_n^{-\delta} \\ &\leq b_{\nu+1}^{-\delta} [1 + B^{-\delta} + B^{-2\delta} + \dots]. \end{aligned}$$

We have

$$(7) \quad |r_\nu| \leq b_{\nu+1}^{-\delta} / (1 - B^{-\delta}) \leq b_\nu^{-\delta} B^{-\delta} / (1 - B^{-\delta}).$$

Since  $|\varphi'| = 2$  we have

$$\varphi(b_n x + b_n h) - \varphi(b_n x) = \pm 2b_n h \quad \text{if } b_n h \leq \frac{1}{2}.$$

So

$$(8) \quad s_\nu = \sum_{n \leq \nu} \pm 2b_n^{1-\delta} h.$$

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Suppose the last term in this sum is positive. Then

$$\begin{aligned} s_\nu &\geq 2h[-b_1^{1-\delta} - \dots - b_{\nu-1}^{1-\delta} + b_\nu^{1-\delta}] \\ &\geq 2hb_\nu^{1-\delta}[-B^{(\nu-1)(\delta-1)} - B^{(\nu-2)(\delta-1)} - \dots - B^{(\delta-1)} + 1] \\ &> 2hb_\nu^{1-\delta} \left[ 1 - \sum_{n=1}^{\infty} B^{(\delta-1)n} \right] \\ &= 2hb_\nu^{1-\delta} [2 - (1 - B^{\delta-1})^{-1}]. \end{aligned}$$

If the last term in (8) is negative, we would have  $s_\nu$  less than the negative of the last expression above. In any case  $|s_\nu| \geq 2hb_\nu^{1-\delta} [2 - (1 - B^{\delta-1})^{-1}]$ .

Combining this with (7) and letting  $h_\nu = (2b_\nu)^{-1}$ , we have

$$|\Delta f| \geq |s_\nu| - |r_\nu| \geq 2^\delta [2 - (1 - B^{\delta-1})^{-1} - B^{-\delta}(1 - B^{-\delta})^{-1}] h_\nu^\delta.$$

The expression in brackets is positive if  $B$  is sufficiently large. This is accomplished by choosing  $b_1$  large. Since  $h_\nu$  is arbitrarily small, the lemma is proved.

Lemmas 3 and 4 and relation (2) yield the following.

**THEOREM 1.** *For the function  $f$ ,  $\dim_s(f) = 1/\delta$ .*

**LEMMA 5.** *For the function  $f$ ,  $V_p^w(f) = 0$  for every  $p > 1/(2 - d)$ .*

**PROOF.** From (8),

$$\begin{aligned} |s_\nu| &\leq \sum_{n \leq \nu} 2b_n^{1-\delta} h \\ &\leq 2hb_\nu^{1-\delta} [1 + B^{\delta-1} + B^{2(\delta-1)} + \dots + B^{(\nu-1)(\delta-1)}] \\ &\leq K_1 hb_\nu^{1-\delta}. \end{aligned}$$

Combining this with (7) yields

$$(9) \quad |\Delta f| \leq |s_\nu| + |r_\nu| \leq K_1 hb_\nu^{1-\delta} + K_2 b_{\nu+1}^{-\delta}.$$

Similarly we obtain

$$(10) \quad \omega(f; x, x + h) \leq K_1 hb_\nu^{1-\delta} + K_2 b_{\nu+1}^{-\delta}$$

by applying (9) to the interval whose endpoints are where  $f$  attains its maximum and minimum in  $[x, x + h]$ .

Now let  $h_\nu = b_\nu^{\delta-1} b_{\nu+1}^{-\delta}$ . Partitioning  $[0, 1]$  into intervals of length  $h_\nu$ , the estimate of  $V_{1/(2-d)}^w(f)$  is (for cases (i) and (ii))

$$\begin{aligned} h_\nu^{-1} [\omega(f; x, x + h_\nu)]^{1/(2-d)} &\leq h_\nu^{-1} [K_1 b_\nu^{1-\delta} h_\nu + K_2 b_{\nu+1}^{-\delta}]^{1/(2-d)} \\ &\leq K_3 b_\nu^{1-\delta} b_{\nu+1}^{\delta(1-d)/(2-d)} \leq K_3. \end{aligned}$$

Now if  $p > 1/(2 - d)$ ,

$$V_p^w(f) \leq h_\nu^{-1} [\omega(f; x, x + h_\nu)]^p \leq K_3 [\omega(f; x, x + h_\nu)]^{p-1/(2-d)},$$

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which converges to zero as  $h_\nu$  goes to zero. For case (iii) repeat the above

argument replacing  $d$  by  $d_n$ . Then if  $p > 1$ ,  $p > 1/(2 - d_n)$  for some  $n$ , and we again obtain  $V_p^w(f) = 0$ .

Lemma 5 and relation (3) yield the following.

**THEOREM 2.** *For the function  $f$ ,  $\dim_w(f) = 1/(2 - d)$ .*

Note that as  $d$  varies from 1 to  $2 - \delta$ ,  $\dim_w(f)$  varies from 1 to  $1/\delta$ . In view of relations (2) and (3), Theorems 1 and 2 show that  $f$  has the largest possible strong variation dimension and the smallest possible weak variation dimension.

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